Final Exam

Math 253, Spring 2021

Name: _____

Tell me what you know: If you don't know the entire answer, showing a formula or writing something illustrating that you understand any concept involved in the problem will allow me to give partial credit. I have to give you a 0 if you write nothing down. You will earn more points by writing down one true thing and no nonsense things than you will by writing down one true thing and a bunch of nonsense things.

Show your work: If you use a result from the textbook that we haven't learned in class, tell me where you found it in the textbook. If you do partial fraction decomposition, show how you solved for your constants. This exam is not testing to see if you can get the right answers. It's testing to see if you know calculus.

Check your answers: Take the time before you turn in your test to make sure you have read the directions correctly and in their entirety, that all your work shown is correct, and that you have clearly stated your answer (by boxing or circling it where appropriate).

Pace yourself: If you're stuck on a problem, move on and come back to it later. Don't risk forcing yourself to give partial answers if you run out of time near the end of the test. Do the easy ones first. There are 94 points on this exam. That means you should budget about 1.1 minute(s) for each point a problem is worth in order to complete the exam in time.

Reminder: The only internet resource that you are allowed to use in conjunction with the exam are the notes that I have written and posted to Canvas. If you use any other online resources or if you communicate with other individuals during the exam, you will receive a zero on the exam and you will be reported to the university for academic misconduct.

1. **Vocabulary** (5 points each)

   (a) Consider the series $\sum_{j=1}^{\infty} \frac{(-1)^j}{j!}$

   i. What is the 3rd **term** in this series?

$$\frac{-1}{6}$$

   ii. What is the 3rd **partial sum** of this series?

$$-1 + \frac{1}{2} - \frac{1}{6}$$

   (b) Suppose $\{c_n\}_{n=1}^{\infty}$ is a sequence and L is a real number. What does it mean for $\lim_{n\to\infty} c_n = L$?

For every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$, $|c_n - L| < \varepsilon$

2. Give an example of a sequence (in either form) which converges to -2 and is neither eventually decreasing nor eventually increasing. If no such sequence exists, explain why.

$$a_n = -2 + \frac{(-1)^n}{n} \quad \text{for } n \geq 1$$

3. Give an example of a power series which has interval of convergence equal to $(-1, 1]$. If no such power series exists, explain why.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n}$$

4. The sequence defined by $a_n = 3 - \frac{2}{n}$ for $n \geqslant 1$ converges to 3 as n approaches infinity. Find an integer n_0 so that $|a_n - 3| < 10^{-5}$ for all $n \geqslant n_0$.

4 [(LHS:) $|a_n - 3| = \left|3 - \frac{2}{n} - 3\right| = \left|-\frac{2}{n}\right| = \frac{2}{n}$

4 [$\frac{2}{n} < 10^{-5} \longleftrightarrow 2 \cdot 10^5 < n$

2 [Take $n_0 = 200001$

5. Does the series $\sum_{k=0}^{\infty} \frac{\cos(k)}{2^k}$ converge or diverge?

Note: $0 \leq \left| \frac{\cos(k)}{2^k} \right| \leq \frac{1}{2^k}$ for $k \geq 0$

$\sum_{k=0}^{\infty} \frac{1}{2^k}$ is a convergent geometric series with $r = \frac{1}{2} < 1$

By the direct comparison test, $\sum_{k=0}^{\infty} \left| \frac{\cos(k)}{2^k} \right|$ converges

Hence, by the absolute convergence test, $\sum_{k=0}^{\infty} \frac{\cos(k)}{2^k}$ converges

6. The series $\sum_{m=2}^{\infty} \frac{(-1)^{m+1}}{\sqrt{m}}$ converges by the Alternating Series Test (you may assume this fact—you don't have to do any work to show this). Find an integer value of k so that S_k (the kth partial sum) is within 10^{-5} of the actual sum.

Want: $|R_k| < 10^{-5}$

4 { Know: $|R_k| \leq \left| \frac{(-1)^{k+2}}{\sqrt{k+1}} \right| = \frac{1}{\sqrt{k+1}}$

4 { $\frac{1}{\sqrt{k+1}} < 10^{-5} \iff 10^5 < \sqrt{k+1}$

$\iff 10^{10} - 1 < k$

2 { Take $\boxed{k = 10^{10}}$

7. Let $r(x) = \dfrac{7}{(x-2)(1+3x)}$.

(a) Find a power series which converges to $r(x)$ centered at $a = 0$.

4

$$\frac{7}{(x-2)(1+3x)} = \frac{A}{x-2} + \frac{B}{1+3x} \rightarrow 7 = A(1+3x) + B(x-2)$$

$x = 2 \rightarrow 7 = A \cdot 7 \rightarrow A = 1$

$x = -\frac{1}{3} \rightarrow 7 = B \cdot \left(-\frac{7}{3}\right)$

$\rightarrow B = -3$

2 $$r(x) = \frac{1}{x-2} - \frac{3}{1+3x} = \frac{1}{-2}\frac{1}{1-\frac{x}{2}} - 3 \cdot \frac{1}{1-(-3x)}$$

2 $$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n - 3\sum_{n=0}^{\infty}(-3x)^n$$

2 $$= \sum_{n=0}^{\infty}\left(-\frac{1}{2^{n+1}} - 3^{n+1}(-1)^n\right)x^n$$

(b) Give an interval of x-values on which your series from part (a) converges. Justify your answer. Note that I am *not* asking for the interval of convergence here.

3 $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$ converges iff $\left|\frac{x}{2}\right| < 1$

$\leftrightarrow x \in (-2, 2)$

3 $\sum_{n=0}^{\infty} (-3x)^n$ converges iff $|-3x| < 1$

$\leftrightarrow x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$

4 Hence, $\sum_{n=0}^{\infty} \left(\frac{-1}{2^{n+1}} - 3^{n+1}(-1)^n\right) x^n$

converges for all $x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$

8. Let $f(x) = 10^x$. Recall that $\frac{d}{dx}(b^x) = b^x \cdot \ln(b)$

(a) Find a formula for $f^{(n)}(x)$.

1 $f'(x) = \ln(10) \cdot 10^x$

1 $f''(x) = \ln(10)^2 \cdot 10^x$

2 $f^{(n)}(x) = \ln(10)^n \cdot 10^x$

(b) Find a formula for the nth Taylor coefficient for $f(x)$ centered at $a = 2$.

$$c_n = \frac{f^{(n)}(2)}{n!} = \frac{\ln(10)^n \cdot 100}{n!}$$

2 - coeff

2 - formula

(c) Use your answer from part (b) to write down the Taylor series for $f(x)$ centered at $a = 2$.

$$\sum_{n=0}^{\infty} \frac{\ln(10)^n \cdot 100}{n!} (x-2)^n$$

2 2

9. The Taylor series for $h(x) = \frac{1}{x^2}$ centered at $a = 1$ is $\sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n$.

(a) Using the definition of $R_n(x)$, compute $R_2(3)$ exactly.

$= \frac{1}{9} - (1 - 4 + 12) = \frac{1}{9} - 7 = -\frac{62}{9}$

$R_2(3) = h(3) - P_2(3) = \frac{1}{9} - \left(1 - 2(3-1) + 3(3-1)^2\right)$

Computing $P_2(3)$ -3

2 4

(b) Using Taylor's Theorem with Remainder, find an upper bound for $|R_2(3)|$.

If: $|h^{(3)}(z)| \le M$ for $z \in [1, 3]$, then

$|R_2(3)| \le \frac{M}{3!}|3-1|^3$ 2

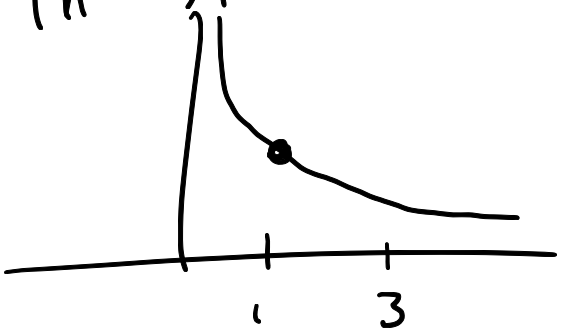
$x = 3$
$a = 1$
$I = [1, 3]$
$k = 2$ 2

2

$h'(x) = -2x^{-3}$ $h''(x) = 6x^{-4}$ $h'''(x) = -24x^{-5}$

2

$|h^{(3)}(x)| \le 24 = M$ 2



1

$\longrightarrow |R_2(3)| \le \frac{24}{3!} \cdot 2^3$

1